ON A WEAK SUM THEOREM IN DIMENSION THEORY*

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ABSTRACT

A metric space $X = \bigcup_{i=0}^{\infty} X_i$ is constructed such that $X_0 = \{x_0\}$ consists of a single point x_0, X_i, $i = 0, 1, 2, \cdots$ are disjoint and closed, X_i, $i = 1, 2, \cdots$ are open, ind $X_i = 0$ for $i = 0, 1, \cdots$ and ind $X = 1$. The above space (proved to be, in some sense, most simple) shows also that the dimension ind of a metric space can be raised by adjoining of a single point, a fact proved recently by E.K. Van Douwen and by T. Przymusiński. Some maximality property of the family $\{X; \operatorname{Ind} X = 0\}$ is proved and conditions implying P-ind $= P$-Ind are given.

1. Introduction

A family F of topological spaces will be said to satisfy wst (the weak sum theorem) if $X \in F$ whenever $X = \bigcup_{i=0}^{\infty} X_i$, where X_i are disjoint closed subsets of X and $X_i \in F, i = 0, 1, \cdots$. A dimension function d will be said to satisfy wst if $F = \{X; d(X) \leqq n\}$ satisfies wst, for $n \geqq -1$.

In this note a metric space $X = \bigcup_{i=0}^{\infty} X_i$ is constructed where X_i, $i = 0, 1, \cdots$ are disjoint closed subsets of X and X_i, $i = 1, 2, \cdots$ are also open in X such that $X_0 = \{x_0\}$ is a one-point set, Ind $X_i = 1$ for $i = 1, 2, \cdots$, ind $X_i = 0$ for $i = 0, 1, \cdots$ and ind $X = 1$. Thus the small inductive dimension $d =$ ind does not satisfy wst. The above space X is also an example of a metric space showing that the dimension ind can be raised by adjoining of a single point x_0, a fact proved recently by Eric K. Douwen in [7]** and by T. Przymusiński in [8]. As known, the dimension functions dim and Ind do satisfy quite strong sum theorems. It will be shown also that the metric space X constructed by us is in some sense the most simple one. It will be proved further that the family $\{X; \operatorname{Ind} X = 0\}$ contains every subfamily of $\{X; \operatorname{ind} X = 0\}$ satisfying wst. Finally, the property wst will be used to give conditions under which P-ind $= P$-Ind (for definitions of P-ind and P-Ind see [4] and [1] respectively; also [5], p. 326).

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In what follows only metric spaces will be considered.

2. An example of a metric space X for which $d = \text{ind}$ does not satisfy wst.

To construct our example we shall use the following result of P. Roy [6].

(1) There exists a metric R such that $\text{Ind}\, R = 1$ and $\text{ind}\, R = 0$.

Now take the space R constructed by P. Roy and put $X = \bigcup_{n=0}^{\infty} X_n$ where $X_n = R \times \{1/n\}$ for $n = 1, 2, \cdots$ and $X_0 = \{x_0\}$ consists only of one point x_0 such that $x_0 \notin \bigcup_{n=1}^{\infty} X_n$. We define the topology in X as follows:

By $\text{Ind}\, R = 1$ there exists a closed set $A \subset R$ and an open set U such that $A \subset U \subset R$ and such that

(2) for every open subset V satisfying $A \subset V \subset U$, V is not closed.
closed.

Since R is a metric space there exists a sequence $\{U_k\}_{k=1,2,\cdots}$ of open sets satisfying:

(3) $$U_1 = U,\ \bar{U}_{k+1} \subset U_k \text{ and } \bigcap_{k=1}^{\infty} U_k = A.$$

We denote $A_n = A \times \{1/n\}$, $U_k^n = U_k \times \{1/n\}\, n, k = 1, 2, \cdots$ and $W_k = \{x_0\} \cup (\bigcup_{n=k}^{\infty} U_k^n)$. Let $B_0 = \{W_k\}_{k=1,2,\cdots}$ and let for $n > 0, B_n$ denote a σ-locally finite base in X_n. Put $B = \bigcup_{n=0}^{\infty} B_n$. Then as trivially seen B is a base for a topology τ in X and we take X with this topology. Let us note the following properties of the topological space X:

(4a) Each $X_n, n = 1, 2, \cdots$ is a homeomorphic copy of R and is a closed and open subset of X,

(4b) X is metrizable.

Indeed, (4a) being evident it suffices by the Nagata-Smirnov theorem to show that X is a regular (Hausdorff) space and that B is a σ-locally finite base.

To show that X is regular we note first that $\bar{W}_{k+1} \subset W_k$ for $k \geqq 1$. In fact, if $x \notin W_k$ then for some $n > 0$ one has $x \in X_n$. If $n \leqq k$ then $X_n \cap W_{k+1} = \varnothing$ and since X_n is open in X one gets $x \notin \bar{W}_{k+1}$. If $n \geqq k + 1$ then by $x \notin W_k$ one has $x \notin U_k^n$. Now by (3) $\bar{U}_{k+1}^n \subset U_k^n$ and so there exists an open set G with $x \in G \subset X_n$. Thus $G \cap U_{k+1}^n = \varnothing$. Hence also $G \cap W_{k+1} = \varnothing$ and again $x \notin \bar{W}_{k+1}$. Now let H be an arbitrary open subset of X and let $x \in H$. If $x \neq x_0$ then $x \in X_n$ for some $n > 0$. Thus $x \in H \cap X_n$ and since X_n is a metric space, there exists an open (in X_n and so by (4a) also in X) set G such that $x \in G \subset \bar{G} \subset H \cap X_n \subset$

H. (Note that since X_n is also closed in X the closure $\bar{G}$ of G in X_n coincides with that in X_n.)

If $x = x_0$ then $x \in W_k \subset H$ for some $k > 0$. But then as already noted $x \in W_{k+1} \subset \bar{W}_{k+1} \subset W_k \subset H$. We thus proved that X is regular.

The fact that X is a Hausdorff space is trivial.

It remains to show that B is a σ-locally finite base. But this is quite evident. Indeed, each B_n can be written in the form $B_n = \bigcup_{i=1}^{\infty} B_n^i$ where each B_n^i is locally finite. Arranging the double sequence $\{B_n^i\}_{n,i=1,2,\cdots}$ into a sequence $\{C_k\}_{k=1,2,\cdots}$ and adding to each C_k one set W_k, i.e. putting $C'_k = C_k \cup \{W_k\}$, one obtains that $B = \bigcup_{k=1}^{\infty} C'_k$ where each C'_k is locally finite. Thus (4b) is proved.

We prove now

THEOREM 1. *The space* $X = \bigcup_{n=0}^{\infty} X_n$ *is a metric space such that* $X_0 = \{x_0\}$ *is a one-point set,* $X_n, n = 1, 2, \cdots$ *are closed and open in* $X, X_n, n = 0, 1, \cdots$ *are disjoint,* $\operatorname{ind} X_n = 0, \operatorname{Ind} X_n = 1$ *for* $n = 1, 2, \cdots$ *and* $\operatorname{ind} X = 1$. (*Thus* $d = \operatorname{ind}$ *does not satisfy* wst.)

PROOF. By the definition of X and by (4a) and (4b) it suffices to show that $\operatorname{ind}_{x_0} X = 1$. (It is trivial that x_0 is the only point at which $\operatorname{ind} X = 1$.) Indeed, suppose to the contrary that $\operatorname{ind} X = 0$. Since $x_0 \in W_1$ there exists then a closed and open set G such that $x \in G \subset W_1$. Since G is open, there exists n such that $x_0 \in W_n \subset G$. Then $A_n \subset U_n^n \subset G \cap X_n \subset U_1^n$. But $G \cap X_n$ is closed and open in X_n contradicting (3), (2) and the definition of A_n and of U_1^n.

REMARK. It is easily seen that $\operatorname{ind} \bigcup_{n=1}^{\infty} X_n = 0$ and so by $X = \{x_0\} \cup (\bigcup_{n=1}^{\infty} X_n)$, one gets that the small inductive dimension ind of a metric space can be raised by adjoining of a single point. Let us also note that our space X satisfying as proved in Theorem 1 $\operatorname{ind} X = 1$ can be represented as a union $X = A \cup B$, where A and B are closed in X with $\operatorname{ind} A = \operatorname{ind} B = 0$. This can be done exactly as in [7] or directly by putting $A = X \setminus \bigcup_{k=1}^{\infty} (W_{4k} \setminus \bar{W}_{4k+2})$ and $B = X \setminus \bigcup_{k=1}^{\infty} (W_{4k+2} \setminus \bar{W}_{4k+4})$. Then since for each i, $W_i \setminus \bar{W}_{i+2}$ is open in X, the sets A and B are closed in X. Also $\operatorname{ind} A = 0$ since for every k the boundary $B(W_{4k+1})$ of W_{4k+1} is contained in $W_{4k} \setminus \bar{W}_{4k+2}$ and so $B(W_{4k+1}) \cap A = \emptyset$. Hence, $W_{4k+1} \cap A$ is a closed and open (in A) neighborhood of x_0. Similarly one shows that $\operatorname{ind} B = 0$.

3. A property of the X constructed in Section 2

In this section we shall show that the space X constructed in Section 2 is in some sense the most simple one. For this purpose we shall need the following:

LEMMA.[†] *Let* $X_i, i = 0, 1, \cdots$ *be metric spaces satisfying* $\operatorname{Ind} X_i = 0$ *for* $i = 1, 2, \cdots$ *and let* $\operatorname{ind} X_0 = 0$. *Let* $X = \bigcup_{i=0}^{\infty} X_i$ *be a metric space such that* $X_i, i = 0, 1, \cdots$ *are closed in* X. *Then* $\operatorname{ind} X = 0$.

PROOF. The proof is similar to the proof of the sum theorem in [3, p. 14]. Suppose that $x \notin X_0$ and let $x \in U$ where U is open in X. Since X_0 is closed, there exists an open set W such that $x \in W \subset \bar{W} \subset U$ and $\bar{W} \cap X_0 = \emptyset$. By the sum theorem for Ind one has $\operatorname{Ind}(\bigcup_{i=1}^{\infty} X_i) = 0$ and thus also $\operatorname{Ind}(X \backslash X_0) = 0$. Hence there exists a closed and open (in $X \backslash X_0$) set V such that $x \in V \subset W \subset \bar{W} \subset U$. The set V is open in X, since $X \backslash X_0$ is open in X. Since $\bar{W} \cap X_0 = \emptyset$ one has also $\bar{V} \cap X_0 = \emptyset$ and so V is also closed in X.

Suppose now that $x \in X_0$ and let F be a closed subset of X such that $x \notin F$. It suffices to show that there exists a closed and open subset U of X such that $x \in U \subset X \backslash F$. Now, by $\operatorname{ind} X_0 = 0$ there exists U_0 closed and open in X_0 such that $x \in U_0 \subset X_0 \backslash F$. The sets $F \cup (X_0 \backslash U_0)$ and U_0 being closed and disjoint, there exist open subsets V_0 and W_0 of X such that $U_0 \subset V_0$, $F \cup (X_0 \backslash U_0) \subset W_0$ and $\bar{V}_0 \cap \bar{W}_0 = \emptyset$. Since $\operatorname{Ind} X_1 = 0$, there exists a set U_1 open and closed in X_1 such that $X_1 \cap \bar{V}_0 \subset U_1 \subset X_1 \backslash \bar{W}_0 \subset X_1 \backslash F$. The sets $\bar{V}_0 \cup U_1$ and $\bar{W}_0 \cup (X_1 \backslash U_1)$ being disjoint and closed in X there exist open subsets V_1 and W_1 of X such that $x \in \bar{V}_0 \cup U_1 \subset V_1$, $F \cap (X_0 \cup X_1) \subset \bar{W}_0 \cup (X_1 \backslash U_1) \subset W_1$ and $\bar{W}_1 \cap \bar{V}_1 = \emptyset$. Proceeding in the same manner one obtains by an easy induction two increasing sequences $V_0, V_1, V_2, \cdots$ and $W_0, W_1, W_2, \cdots$ of open sets such that for $V = \bigcup_{i=0}^{\infty} V_i$ and $W = \bigcup_{i=0}^{\infty} W_i$ one has $V \cap W = \emptyset$ and $V \cup W = X$. The sets V and W being open and disjoint are also closed in X and satisfy $x \in V$ and $F \subset W$. Thus $\operatorname{ind} X = 0$.

The following theorem shows that in some sense the space X constructed in Section 2 is most simple.

THEOREM 2. *Let* $X = \bigcup_{i=1}^{\infty} X_i$ *be a metric space, where* X_i *are disjoint and closed subsets of* X *satisfying* $\operatorname{ind} X_i = 0$ *for* $i = 1, 2, \cdots$. *If* $\operatorname{ind} X > 0$ *then for infinitely many indices* i *one has* $\operatorname{Ind} X_i > 0$.

PROOF. Suppose to the contrary that there exists n such that $\operatorname{Ind} X_i = 0$ for all $i > n$ and put $X' = \bigcup_{i=1}^{n} X_i$. Since X_i are closed and disjoint one has $\operatorname{ind} X' = 0$. Applying the lemma one obtains that $\operatorname{ind} X = 0$, contradicting $\operatorname{ind} X > 0$.

4. Property wst and the family $\{X; \operatorname{Ind} X = 0\}$

Theorem 3 which will be proved in this section shows that the family

† A sorter *proof* (using the sum theorem for Ind) has recently been communicated to me by E. K. van Douwen.

$\{X; \operatorname{Ind} X = 0\}$ contains every subfamily of the family $\{X; \operatorname{ind} X = 0\}$ satisfying wst. Let us recall that we consider only metric spaces. We introduce the following:

DEFINITION. Let Q be a non-empty topologically closed and monotone family of spaces. Put $d_Q(X) \leqq n$ if and only if there exist $n+1$ subspaces X_i of X such that $X_i \in Q, i = 1, 2, \cdots n+1$ and $X = \bigcup_{i=1}^{n+1} X_i$. For example, if $Q = \{X; \operatorname{Ind} X = 0\}$ then $d_Q(X) = \operatorname{Ind} X$.

Note that because of the monotonicity of Q one has $d_Q(Y) \leqq d_Q(X)$ for $Y \subset X$, i.e. d_Q is monotone.

Let us denote $S = \{X; \operatorname{ind} X = 0\}$.

THEOREM 3. *If Q is a subfamily of S satisfying* wst *then* $Q \subset \{X; \operatorname{Ind} X = 0\}$ (*thus* $\operatorname{Ind} X \leqq d_Q(X)$).

PROOF. Suppose that $R \in Q$. If there would be $\operatorname{Ind} R > 0$ then defining $X_n, n = 0, 1, \cdots$ and X as in Section 2 (note that Q is topologically closed and monotone, thus the onepoint set $X_0 = \{x_0\}$ belongs to Q) one obtains as in Theorem 1 that the metric space $X = \bigcup_{n=0}^{\infty} X_n$ satisfies $\operatorname{ind} X > 0$, where $X_n \in Q$ are closed and disjoint subsets of $X, n = 0, 1, \cdots$. Since $Q \subset S$ one has $X \notin Q$. But then Q does not satisfy wst. Thus $R \in Q$ implies $\operatorname{Ind} R = 0$.

5. Property wst and the equality of P-ind and P-Ind

In this section property wst and the equality $P\text{-ind} = P\text{-Ind}$ will be investigated. We recall first the following:

DEFINITION ([1] and [4]). Let P be a non-empty topologically closed family of metric spaces. We put $P\text{-ind}\, X = -1$ ($P\text{-Ind}\, X = -1$) if and only if $X \in P$.

We define $P\text{-ind}\, X \leqq n$ ($P\text{-Ind}\, X \leqq n$) if and only if for every $x \in X$ (for every closed subset A of X) there exist arbitrarily small neighbourhoods U of x (of A) such that $P\text{-ind}\, B(U) \leqq n-1$ ($P\text{-Ind}\, B(U) \leqq n-1$). Finally, we put P-ind $X = n$ ($P\text{-Ind}\, X = n$) if $P\text{-ind}\, X \leqq n$ ($P\text{-Ind}\, X \leqq n$), but (P-Ind $X \leqq n-1$) does not hold.

THEOREM 4. *If P is a non-empty topologically closed family of metric spaces such that closed subsets of spaces belonging to P also belong P (i.e. P is monotone relative to closed subsets or closed monotone) and if P*-ind *satisfies* wst *then for every metric space X one has P*-ind $X = P$-Ind X.

PROOF. By the definition of P-ind X and P-Ind X one has P-ind $X \leqq P$-Ind X. Now, suppose to the contrary that there exists a metric space R such that P-ind $R < P$-Ind R. We shall show by induction on P-ind R that P-ind does not satisfy wst. We note first that P-ind $X = -1$ if and only if P-Ind $X = -1$. Suppose that P-ind $R = 0$ and P-Ind $R > 0$. Define the space $X = \bigcup_{n=0}^{\infty} X_n$ as in Section 2. Note that P-ind $X_0 = P$-ind$\{x_0\} \leqq 0$. Then P-ind $X_n = 0$ for $n = 0, 1, \cdots$ and, as easily seen, P-ind $X > 0$ (thus P-ind does not satisfy wst). Indeed one has P-ind $X \geqq 0$. Otherwise $X \in P$ and thus, since X_n for $n > 0$ is homeomorphic to R and is closed in X, one gets by the monotonicity of P (relative to closed subsets) that $R \in P$, contradicting P-ind $R = 0$.

Let us therefore assume that P-ind $X = 0$. Then as in the proof of Theorem 1 one has for sufficiently small neighborhoods G of x_0 that $B(G) \in P$. Since P is monotone relative to closed subsets (i.e., P is closed monotone), it follows that $B(G) \cap X_n = B(G \cap X_n) \in P$ for $n > 0$, contradicting (as in the proof of Theorem 1) the fact that P-Ind $R > 0$. It follows that P-ind $X > 0$, contradicting the assumption that P-ind satisfies wst. Suppose now inductively that if for $k \leqq n$ there exists a space R_k such that $k = P$-ind $R_k < P$-Ind R_k, then P-ind does not satisfy wst.

Let R be a space such that $n + 1 = P$-ind $R < P$-Ind R. Define X as in Section 2. By [4, Theorem 3.3] P-ind is monotone relative to closed subsets. Hence exactly as before (in the case P-ind $R = 0$) one obtains for sufficiently small neighborhoods G of x_0 that P-Ind $B(G) \geqq n + 1$. If also P-ind $B(G) \geqq n + 1$ for every sufficiently small neighbourhood G of x_0, then P-ind $X \geqq n + 2$ and the theorem holds. Otherwise there exists $k \leqq n$ such that

$$k = P\text{-ind}\, B(G) < P\text{-Ind}\, B(G)$$

and again the theorem holds by the induction assumption.

REFERENCES

1. J. M. Aarts, *A characterization of strong inductive dimension*, Fund. Math. **70** (1971), 147–155.

2. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1941.

3. J. I. Nagata, *Modern Dimension Theory*, Amsterdam, 1965.

4. T. Nishiura, *Inductive invariants and dimension theory*, Fund. Math. **59** (1966), 243–262.

5. M. Reichaw, *On a theorem of W. Hurewicz on mappings which lower dimension*, Colloq. Math. **26** (1972), 323–329.

6. P. Roy, *Nonequality of dimensions for metric spaces*, Trans. Amer. Math. Soc. **134** (1968), 117–132.

7. E. K. Van Douwen, *The small inductive dimension can be raised by adjunction of a single point*, Indag. Math. **5** (1973), 434–442.

8. T. Przymusinski, *A note on dimension theory of metric spaces*, Fund. Math. **85** (1974), 272–284.

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